Equations in divergence form notes

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1 Definitions and notation

Let Ω be a domain in $\mathbb{R}^n$. We say that $u \in W^{1,2}(\Omega)$ satisfies

$$Lu = D_i(a^{ij} D_j u + b^i u) + c^j D_j u + du = D_i f^i + g \text{ weakly in } \Omega, \tag{1}$$

where the coefficients a^{ij}, b^i, c^j, and d are measurable functions on Ω and $f^i, g \in L^2(\Omega)$, if

$$\int_\Omega ((a^{ij} D_j u + b^i u) D_i \zeta - (c^j D_j u + du)\zeta) = \int_\Omega (f^i D_i \zeta - g\zeta) \tag{2}$$

for all *test functions* $\zeta \in C^\infty_c(\Omega)$. We call u a *weak solution* to (1). Note that if (2) holds true for all $\zeta \in C^\infty_c(\Omega)$, then by a continuity argument using $C^\infty_c(\Omega)$ being dense in $W^{1,2}_0(\Omega)$, (2) holds true for all $\zeta \in W^{1,2}_0(\Omega)$

Observe that if the functions u, a^{ij}, b^i, c^j, d, f^i, and g were sufficiently smooth on Ω, for example $u \in C^2(\Omega)$, $a^{ij}, b^i, f \in C^1(\Omega)$, and $c^j, d, g \in C^0(\Omega)$, then by integration by parts,

$$Lu = D_i(a^{ij} D_j u + b^i u) + c^j D_j u + du = D_i f^i + g \text{ pointwise in } \Omega \tag{3}$$

implies that (2) holds true and conversely (2) implies that

$$-\int_\Omega Lu\zeta = -\int_\Omega (D_i f^i + g)\zeta$$

for all $\zeta \in C^\infty_c(\Omega)$, which since ζ is arbitrary implies (3). However, (3) does not make sense under the weaker regularity conditions that $u \in W^{1,2}(\Omega)$, a^{ij}, b^i, c^j, and d are measurable functions on Ω, and $f^i, g \in L^2(\Omega)$, whereas (2) does make sense under the weaker regularity conditions.

We shall assume the ellipticity condition

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for a.e. } x \in \Omega \text{ and for all } \xi \in \mathbb{R}^n \tag{4}$$

for some constant $\lambda > 0$. Note that for equations in divergence form we cannot assume that $a^{ij}(x) = a^{ji}(x)$ for a.e. $x \in \Omega$. It will be standard to assume that the coefficients are bounded with

$$\sum_{i,j=1}^n |a^{ij}(x)|^2 \leq \Lambda^2, \quad \lambda^{-2}\sum_{i=1}^n (|b^i(x)|^2 + |c^i(x)|^2) + \lambda^{-1}|d^i(x)| \leq \nu^2 \text{ for a.e. } x \in \Omega \tag{5}$$

for some constants $\Lambda, \nu \in (0, \infty)$.

We can similarly consider differential inequalities

$$Lu = D_i(a^{ij} D_j u + b^i u) + c^j D_j u + du \geq (\leq) D_i f^i + g \text{ weakly in } \Omega$$

for $u \in W^{1,2}(\Omega)$, which we take to mean that

$$\int_\Omega ((a^{ij} D_j u + b^i u) D_i \zeta - (c^j D_j u + du)\zeta) \leq (\geq) \int_\Omega (f^i D_i \zeta - g\zeta) \tag{6}$$

for all non-negative $\zeta \in C_c^\infty(\Omega)$ (or equivalently for all $\zeta \in W_0^{1,2}(\Omega)$).

We also want to consider the Dirichlet problem

$$\begin{aligned} D_i(a^{ij} D_j u + b^i u) + c^j D_j u + du &= D_i f^i + g \text{ weakly in } \Omega, \\ u &= \varphi \text{ on } \partial\Omega, \end{aligned}$$

where $u \in W^{1,2}(\Omega)$, the coefficients a^{ij}, b^i, c^j, and d are bounded measurable functions on Ω, $f^i, g \in L^2_{\text{loc}}(\Omega)$, and $\varphi \in W^{1,2}(\Omega)$. By $u = \varphi$ on $\partial\Omega$, we mean that

$$u - \varphi \in W_0^{1,2}(\Omega).$$

Note that if Ω, u, and φ are sufficiently smooth, namely Ω is a C^1 domain and $u, \varphi \in C^1(\overline{\Omega})$, then $u - \varphi \in W_0^{1,2}(\Omega)$ implies that $u = \varphi$ pointwise on $\partial\Omega$. To see this, recall that $u - \varphi \in W_0^{1,2}(\Omega)$ means that there exists a sequence of functions $v_j \in C_c^\infty(\Omega)$ such that $v_j \to u - \varphi$ in $W^{1,2}(\Omega)$. Thus

$$\begin{aligned} \int_{\partial\Omega} (u - \varphi)\zeta \cdot \nu &= \int_\Omega (D(u - \varphi) \cdot \zeta + (u - \varphi) \operatorname{div} \zeta) \\ &= \lim_{j \to \infty} \int_\Omega (Dv_j \cdot \zeta + v_j \operatorname{div} \zeta) \\ &= \lim_{j \to \infty} \int_{\partial\Omega} v_j \zeta \cdot \nu \\ &= 0, \end{aligned}$$

for all $\zeta \in C_c^\infty(\mathbb{R}^n; \mathbb{R}^n)$, where ν denotes the outward unit normal to $\partial\Omega$. Since ζ is arbitrary, $u = \varphi$ pointwise on $\partial\Omega$.

2 Maximum principle

Let Ω be a bounded domain in $\mathbb{R}^n$. Let $u \in W^{1,2}(\Omega)$. By $\sup_\Omega u$ we mean the essential supremum, i.e.

$$\sup_\Omega u = \inf\{k \in \mathbb{R} : u \leq k \text{ a.e. in } \Omega\}.$$

By $\sup_{\partial\Omega} u$, we mean

$$\sup_{\partial\Omega} u = \inf\{k \in \mathbb{R} : (u - k)^+ \in W_0^{1,2}(\Omega)\},$$

where $v^+(x) = \max\{v(x), 0\}$ for measurable functions v on Ω. Using the fact that $\lim_{\varepsilon \downarrow 0} \mathcal{L}^n(\{x \in \Omega : 0 < u(x) - \sup_{\partial\Omega} u < \varepsilon\}) = 0$ and $W_0^{1,2}(\Omega)$ is closed in $W^{1,2}(\Omega)$, given easy to see that $(u - k)^+$

converges to $(u-\sup_{\partial\Omega} u)^+$ in $W_0^{1,2}(\Omega)$ as $k \downarrow \sup_{\partial\Omega} u$, so in particular $(u-\sup_{\partial\Omega} u)^+ \in W_0^{1,2}(\Omega)$. We can similarly define

$$\inf_{\Omega} u = \sup\{k \in \mathbb{R} : u \geq k \text{ a.e. in } \Omega\},$$
$$\inf_{\partial\Omega} u = \sup\{k \in \mathbb{R} : (u-k)^- \in W_0^{1,2}(\Omega)\}.$$

where $v^-(x) = \min\{v(x), 0\}$ for measurable functions v on Ω. Given $u \in W^{1,2}(\Omega)$, obviously $u \leq v$ means that $u \leq v$ a.e. in Ω. We say $u \leq v$ on $\partial\Omega$ if $(u-v)^+ \in W_0^{1,2}(\Omega)$.

Theorem 1 (Weak maximum principle)**.** *Let Ω be a bounded domain in $\mathbb{R}^n$. Suppose $u \in W^{1,2}(\Omega)$ satisfies*

$$Lu = D_i(a^{ij}D_ju + b^iu) + c^jD_ju + du \geq 0 \text{ in } \Omega$$

where a^{ij}, b^i, c^j, and d are measurable function on Ω satisfying (4) and (5) for some constants $0 < \lambda, \Lambda, \nu < \infty$ and

$$\int_{\Omega} (-b^iD_i\zeta + d\zeta) \leq 0 \tag{7}$$

for all nonnegative $\zeta \in W_0^{1,1}(\Omega)$. Then

$$\sup_{\Omega} u \leq \sup_{\partial\Omega} u^+,$$

where $u^+(x) = \max\{u(x), 0\}$ for $x \in \Omega$.

Heuristically,

$$Lu = D_i(a^{ij}D_ju + b^iu) + c^jD_ju + du = a^{ij}D_{ij}u + (D_ia^i + b^i + c^i)D_iu + (D_ib^i + d)u \text{ in } \Omega.$$

Since if b^i and d are sufficiently smooth ($b^i \in W^{1,1}(\Omega)$ and $d \in L^1(\Omega)$ is sufficient), then by integration by parts $D_ib^i + d \leq 0$ a.e. in Ω is equivalent to (7). Thus we can interpret (7) as meaning that $D_ib^i + d \leq 0$ weakly in Ω. (7) is the analogue to $c \leq 0$ in the case of the classical elliptic operator $Lu = a^{ij}D_{ij}u + b^iD_iu + cu$.

Proof of the weak maximum principle. We will use a standard type of proof technique using the weak inequality

$$\int_{\Omega} ((a^{ij}D_ju + b^iu)D_i\zeta - (c^jD_ju + du)\zeta) \leq 0. \tag{8}$$

for all nonnegative $\zeta \in W_0^{1,2}(\Omega)$.

Our first step it to use (7) to simplify the inequality. By rewriting (8) and using (7),

$$\int_{\Omega} (a^{ij}D_juD_i\zeta - (b^j + c^j)D_ju\zeta) \leq \int_{\Omega} (-b^iD_i(u\zeta) + d(u\zeta)) \leq 0. \tag{9}$$

for all $\zeta \in W_0^{1,2}(\Omega)$ such that $\zeta \geq 0$ and $u\zeta \geq 0$ a.e. in Ω. Note that $u \in W^{1,2}(\Omega)$ and $\zeta \in W_0^{1,2}(\Omega)$ implies that $u\zeta \in W_0^{1,1}(\Omega)$.

The case where $b^j + c^j = 0$ a.e. in Ω is particularly easy. We now will chose a particular test function ζ in (9), namely $\zeta = (u-l)^+$ for $l = \sup_{\partial\Omega} u^+$. Note that this ζ is indeed in $W_0^{1,2}(\Omega)$. By (9) obtain

$$\int_{\Omega} a^{ij}D_j\zeta D_i\zeta \leq 0.$$

By (4),

$$\lambda \int_\Omega |D\zeta|^2 \leq 0,$$

so $D\zeta = 0$ a.e. in Ω. Thus ζ is constant on Ω. In particular, since $\zeta \in W^{1,2}_0(\Omega)$, $\zeta = 0$ a.e. in Ω. Therefore

$$\sup_\Omega u \leq l = \sup_{\partial\Omega} u^+.$$

Now suppose $b^j + c^j$ is not identically zero on Ω. By way of contradiction suppose that

$$\sup_{\partial\Omega} u^+ < \sup_\Omega u.$$

Let $l \in \mathbb{R}$ such that

$$\sup_{\partial\Omega} u^+ < l < \sup_\Omega u.$$

Now we proceed with a standard type of argument. Like before, we choose our test function ζ, in particular we choose $\zeta = (u-l)^+$ in (9). We note that $\zeta \in W^{1,2}_0(\Omega)$. Then by (9)

$$\int_\Omega a^{ij} D_j\zeta D_i\zeta - (b^j + c^j)\zeta D_j\zeta) \leq 0.$$

Next we rewrite this inequality as

$$\int_\Omega a^{ij} D_j\zeta D_i\zeta \leq \int_\Omega (b^j + c^j)\zeta D_j\zeta$$

so that the integral of $a^{ij} D_j\zeta D_i\zeta$ is on the left hand side and everything else is on the right hand side. Then by (4) and (5),

$$\lambda \int_\Omega |D\zeta|^2 \leq 2\lambda\nu \int_\Omega \zeta |D\zeta|.$$

Next we move all the $D\zeta$ terms to the left hand side using the Cauchy inequality $ab \leq \frac{1}{4}a^2 + b^2$ for $a, b \geq 0$ to get

$$\lambda \int_\Omega |D\zeta|^2 \leq \frac{\lambda}{2} \int_\Omega |D\zeta|^2 + 2\lambda\nu^2 \int_\Gamma |\zeta|^2$$

where $\Gamma = \{x \in \Omega : D\zeta(x) \neq 0\}$, and then move the integral of $|D\zeta|^2$ to the left hand side to get

$$\int_\Omega |D\zeta|^2 \leq 4\nu^2 \int_\Gamma |\zeta|^2. \tag{10}$$

Note that here we used the fact that $\zeta|D\zeta| = 0$ on $\Omega \setminus \Gamma$ to get an integral over Γ on the right hand side of (10). This will be important in a moment. The next step is to apply the Sobolev inequality on the left hand side to obtain

$$\frac{1}{C^2} \|\zeta\|^2_{L^{2n/(n-2)}} \leq 4\nu^2 \int_\Gamma |\zeta|^2$$

for some constant $C = C(n) \in (0, \infty)$ and then apply the Hölder inequality to the right hand side to obtain

$$\frac{1}{C^2} \|\zeta\|^2_{L^{2n/(n-2)}} \leq 4\nu^2 |\Gamma|^{2/n} \|\zeta\|^2_{L^{2n/(n-2)}},$$

where $|S|$ denotes the Lebesgue measure of a set S, which by cancelling $\|\zeta\|_{L^{2n/(n-2)}} > 0$ implies

$$(2C\nu)^{-n} \leq |\Gamma|. \tag{11}$$

Now the application of the Sobolev inequality to the left hand side of (10) only makes sense if $n > 2$. If $n = 2$, let $1 < \hat{n} < 2$. and note that by the (10), Sobolev inequality, and the Hölder inequality

$$\begin{aligned}\frac{1}{C}\|\zeta\|_{L^{2\hat{n}/(2-\hat{n})}(\Gamma)} &\leq \|D\zeta\|_{L^{\hat{n}}(\Gamma)} \\ &\leq |\Gamma|^{1/\hat{n}-1/2}\|D\zeta\|_{L^2(\Gamma)} \\ &\leq 2\nu|\Gamma|^{1/\hat{n}-1/2}\|\zeta\|_{L^2(\Gamma)} \quad \text{(by (10))} \\ &\leq 2\nu|\Gamma|^{1/2}\|\zeta\|_{L^{2\hat{n}/(2-\hat{n})}(\Gamma)}\end{aligned}$$

so cancelling $\|\zeta\|_{L^{2\hat{n}/(2-\hat{n})}(\Gamma)}$ yields (11) in the case $n = 2$. Since Γ is where $\zeta = (u - l)^+$ satisfies $\zeta \geq 0$ and $D\zeta \neq 0$, we can rewrite (11) as

$$C \leq |\{x : u(x) \geq l,\ Du(x) \neq 0\}|. \tag{12}$$

Note that the set on the right hand side of (12) is decreasing (with respect to set inclusion $\subseteq$) as l increases. If $\sup_\Omega u = \infty$, let $l \uparrow \infty$ in (12)to obtain $u(x) = \infty$ on a subset of Ω with positive measure, which contradicts $u \in L^2(\Omega)$. If $\sup_\Omega u < \infty$, let l increase to $\sup_\Omega u$ in (12), we obtain $u = \sup_\Omega u$ and $Du \neq 0$ on a subset of Ω of positive measure, which is impossible by Lemma 1 below. □

Lemma 1. *Let $u \in W^{1,2}(\Omega)$. If u is constant on some measurable set S in Ω, then $Du = 0$ a.e. on S.*

Proof. WLOG suppose $u = 0$ on S.

Recall that if $f : \mathbb{R} \to \mathbb{R}$ is a C^1 function with bounded derivative and $u \in W^{1,2}(\Omega)$, then $f(u) \in W^{1,2}(\Omega)$ with weak derivative $f'(u)Du$. We claim that for $u^+(x) = \max\{u(x), 0\}$, $u^+ \in W^{1,2}(\Omega)$ with $Du^+(x) = Du(x)$ at a.e. $x \in \Omega$ with $u(x) > 0$ and $Du^+(x) = 0$ for a.e. $x \in \Omega$ with $u(x) \leq 0$. To see this, for $\varepsilon > 0$ let $f_\varepsilon : \mathbb{R} \to \mathbb{R}$ a smooth, convex function with $f_\varepsilon(t) = 0$ for $t \leq \varepsilon/2$ and $f'_\varepsilon(t) = 1$ for $t \geq \varepsilon$. Fix a test function $\zeta \in C_c^\infty(\Omega)$. Since $f_\varepsilon(t) = 0$ for $t \leq 0$ and $t - \varepsilon < f_\varepsilon(t) \leq t$ for $t \geq 0$,

$$\left|\int_\Omega u^+ D\zeta - \int_\Omega f_\varepsilon(u) D\zeta\right| \leq \varepsilon \int_\Omega |\zeta| \to 0$$

as $\varepsilon \downarrow 0$. Hence

$$-\int_\Omega u^+ D\zeta = \lim_{\varepsilon\downarrow 0} -\int_\Omega f_\varepsilon(u) D\zeta = \lim_{\varepsilon\downarrow 0} \int_\Omega f'_\varepsilon(u) Du\zeta = \int_{\Omega\cap\{u>0\}} Du\zeta,$$

where the last step follows from the dominated convergence theorem and the fact that $f'_\varepsilon(u) = 0$ if $u \leq 0$ and $f'_\varepsilon(u) \uparrow 1$ if $u > 0$.

By the same argument, we can show that for $u^-(x) = \min\{u(x), 0\}$, $u^- \in W^{1,2}(\Omega)$ with $Du^-(x) = Du(x)$ at a.e. $x \in \Omega$ with $u(x) < 0$ and $Du^-(x) = 0$ for a.e. $x \in \Omega$ with $u(x) \geq 0$.

Since $u = 0$ a.e. on S, $Du^+ = Du^- = 0$ a.e. on S. Since $u = u^+ + u^-$, $Du = Du^+ + Du^- = 0$ a.e. on S. □

Corollary 1 (Uniqueness for the Dirichlet Problem)**.** *Consider L as above (i.e., satisfying* (4), (5), *and* (7)*) Suppose $u, v \in W^{1,2}(\Omega)$ such that $Lu = Lv$ in Ω and $u = v$ on $\partial\Omega$ (i.e. $u - v \in W_0^{1,2}(\Omega)$). Then $u = v$ a.e. in Ω.*

3 Existence theory

Recall the following:

Theorem 2 (Lax Milgram). *Let $\mathcal{H}$ be a Hilbert space, $B : \mathcal{H} \times \mathcal{H} \to \mathbb{R}$ be a bilinear functional that is*

Bounded: $|B(x,y)| \leq C_1 \|x\|_{\mathcal{H}} \|y\|_{\mathcal{H}}$ for all $x, y \in \mathcal{H}$ for some constant $C_1 \in (0, \infty)$ and

Coercive: $B(x,x) \geq C_2 \|x\|^2_{\mathcal{H}}$ for all $x \in \mathcal{H}$ for some constant $C_2 \in (0,\infty)$.

Let $F : \mathcal{H} \to \mathbb{R}$ be a bounded linear functional on $\mathcal{H}$. Then there exists a unique element $z \in \mathcal{H}$ such that

$$B(z,x) = F(x) \text{ for all } x \in \mathcal{H}. \tag{13}$$

Moreover, $\|z\|_{\mathcal{H}} \leq (1/C_2)\|F\|$. (References: Gilbarg and Trudinger, Theorem 5.8)

Proof. For every $x \in \mathcal{H}$, by Riesz representation applied to the bounded linear functional $B(x,.)$, there is a unique element $Tx \in \mathcal{H}$ such that $B(x,y) = (Tx,y)_{\mathcal{H}}$ and $\|B(x,.)\|_{\mathcal{H}^*} = \|Tx\|_{\mathcal{H}}$. Since B is bilinear, $T : \mathcal{H} \to \mathcal{H}$ is linear. Also by Riesz representation applied to F, there is a unique $w \in \mathcal{H}$ such that $F(x) = (w,x)$ for all $x \in \mathcal{H}$. Thus (13) is equivalent to

$$(Tz, x) = (w, x) \text{ for all } x \in \mathcal{H}, \tag{14}$$

which is in turn equivalent to

$$Tz = w. \tag{15}$$

((14) implies (15) by choosing $x = Tz - w$.) Thus in order to show that there is a solution z to (13) with $\|z\|_{\mathcal{H}} \leq (1/C_2)\|F\|$, it suffices to show that T has an inverse function $T^{-1} : \mathcal{H} \to \mathcal{H}$ which is a bounded linear map with norm $\|T^{-1}\| \leq 1/C_2$.

Since B is coercive,

$$C_2 \|x\|^2_{\mathcal{H}} \leq B(x,x) = (x, Tx) \leq \|x\|_{\mathcal{H}} \|Tx\|_{\mathcal{H}},$$

so

$$C_2 \|x\|_{\mathcal{H}} \leq \|Tx\|_{\mathcal{H}}. \tag{16}$$

Now we use (16) to show that $T : \mathcal{H} \to \mathcal{H}$ is bijective and $\|T^{-1}\| \leq 1/2C_2$:

(1) T is injective: If $Tx_1 = Tx_2$ for some $x_1, x_2 \in \mathcal{H}$ then $T(x_1 - x_2) = 0$ in $\mathcal{H}$ and by (16) $\|x_1 - x_2\|_{\mathcal{H}} = 0$, so $x_1 = x_2$.

(2) T has closed range: Suppose $x_j \in \mathcal{H}$ such that $Tx_j \to y$ in $\mathcal{H}$. Then by (16)

$$\|x_j - x_k\|_{\mathcal{H}} \leq \frac{1}{C_2} \|Tx_j - Tx_k\|_{\mathcal{H}} \to 0$$

as $j, k \to \infty$, so x_j is Cauchy. Thus x_j converges to some x in $\mathcal{H}$ and $y = Tx$. .

(3) The range $T(\mathcal{H})$ of T is $\mathcal{H}$: Suppose there is a $x \in T(\mathcal{H})^\perp$. By coercivity,

$$C_2 \|x\|^2_{\mathcal{H}} \leq B(x,x) = (Tx, x)_{\mathcal{H}} = 0,$$

so $x = 0$. Therefore $T(\mathcal{H}) = \mathcal{H}$.

(4) $\|T^{-1}\| \leq 1/C_2$: Let $y = T^{-1}x$. Then by (16),

$$\|T^{-1}x\|_{\mathcal{H}} = \|y\|_{\mathcal{H}} \leq \frac{1}{C_2}\|Ty\|_{\mathcal{H}} = \frac{1}{C_2}\|x\|_{\mathcal{H}}.$$

□

Theorem 3. *Let Ω be a domain in $\mathbb{R}^n$. Let $a^{ij}, b^i, c^j, d \in L^\infty(\Omega)$ be coefficients satisfying (4) and (5) for some constants $0 < \lambda, \Lambda, \nu < \infty$ and (7). For every $f^i, g \in L^2(\Omega)$ and $\varphi \in W^{1,2}(\Omega)$, there is a unique solution $u \in W^{1,2}(\Omega)$ to the Dirichlet problem*

$$\begin{aligned} Lu &= D_i f^i + g \text{ weakly in } \Omega, \\ u &= \varphi \text{ on } \partial\Omega. \end{aligned} \tag{17}$$

Moreover,

$$\|u\|_{W^{1,2}(\Omega)} \leq C(\|f\|_{L^2(\Omega)} + \|g\|_{L^2(\Omega)} + \|\varphi\|_{W^{1,2}(\Omega)}).$$

Proof. By the maximum principle, the solution to the Dirichlet problem is unique if it exists, so what remains to show is the existence of solutions. By replacing u with $v = u - \varphi$ and solving for v such that $Lv = D_i f^i + g - L\varphi$ weakly in Ω and $v = 0$ on $\partial\Omega$, it suffices to assume that $\varphi = 0$ a.e. on Ω.

Define the bounded bilinear functional $\mathcal{L} : W_0^{1,2}(\Omega) \times W_0^{1,2}(\Omega) \to \mathbb{R}$ by

$$\mathcal{L}(u, v) = \int_\Omega (a^{ij} D_j u D_i v + b^i u D_i v - c^j D_j u v - duv)$$

and define the bounded linear functional $F : W_0^{1,2}(\Omega) \to \mathbb{R}$ by

$$F(\zeta) = \int_\Omega (f^i D_i \zeta - g\zeta).$$

Clearly solving for $u \in W^{1,2}(\Omega)$ satisfying (17) with $\varphi = 0$ is equivalent to solving for $u \in W_0^{1,2}(\Omega)$ such that

$$\mathcal{L}(u, \zeta) = F(\zeta) \text{ for all } \zeta \in W_0^{1,2}(\Omega). \tag{18}$$

By Lax-Milgram, it suffices to show that $\mathcal{L}$ is coercive. Unfortunately, we only have

$$\begin{aligned} \mathcal{L}(v, v) &= \int_\Omega (a^{ij} D_i v D_j v + (b^i - c^i) v D_i v - dv^2) \\ &\geq \lambda \int_\Omega |Dv|^2 - \lambda \int_\Omega (2\nu |v||Dv| + \nu^2 |v|^2) && \text{(by (4) and (5))} \\ &\geq \frac{\lambda}{2} \int_\Omega |Dv|^2 - 3\lambda\nu^2 \int_\Omega v^2 && \text{(by Cauchy's inequality)}, \end{aligned}$$

so $\mathcal{L}$ is not necessarily coercive.

If we instead considered the problem of solving for $u \in W_0^{1,2}(\Omega)$ such that

$$L_\sigma u \equiv Lu - \sigma u = D_i f^i + g \text{ weakly in } \Omega$$

for given $f^i, g \in W_0^{1,2}(\Omega)$, then the corresponding bilinear form

$$\mathcal{L}_\sigma(u,v) = \int_\Omega (a^{ij} D_j u D_i v + b^i u D_i v - c^j D_j u v + (-d+\sigma) u v)$$

would satisfy

$$\mathcal{L}_\sigma(v,v) \geq \frac{\lambda}{2} \int_\Omega |Dv|^2 + (\sigma - 3\lambda\nu^2) \int_\Omega v^2,$$

so $\mathcal{L}_\sigma$ is obviously coercive provided σ is sufficiently large. By Lax-Milgram, there exists an inverse map $L_\sigma^{-1} : W_0^{1,2}(\Omega)^* \to W_0^{1,2}(\Omega)$ such that for every bounded linear functional $F \in W_0^{1,2}(\Omega)^*$, $u = TF$ is the solution to $L_\sigma u = F$ weakly in Ω, i.e. $\mathcal{L}_\sigma(u,\zeta) = F(\zeta)$ for all $\zeta \in W_0^{1,2}(\Omega)$.

Observe that $Lu = F$ weakly in Ω, where $F \in W_0^{1,2}(\Omega)^*$, is equivalent to

$$u + \sigma L_\sigma^{-1} u = L_\sigma^{-1} F \text{ in } \Omega. \tag{19}$$

We know

$$L_\sigma^{-1} : W_0^{1,2}(\Omega) \subset L^2(\Omega) \subset W_0^{1,2}(\Omega)^* \to W_0^{1,2}(\Omega),$$

which is compact since the embedding $W_0^{1,2}(\Omega) \subset L^2(\Omega)$ is compact by Rellich's lemma. Note that here the embedding $L^2(\Omega) \subset W_0^{1,2}(\Omega)^*$ is defined by mapping $v \in L^2(\Omega)$ to the linear functional $\zeta \mapsto \int_\Omega v\zeta$. Since L_σ^{-1} is a compact linear operator between Banach spaces, by spectral theory for L_σ^{-1} either $-1/\sigma$ is an eigenvalue of L_σ^{-1} or (19) has a unique solution $u \in W_0^{1,2}(\Omega)$ for all $F \in W_0^{1,2}(\Omega)^*$ and $\|u\|_{W^{1,2}(\Omega)} \leq C\|F\|$ for some $C = C(\lambda, \Lambda, \nu) \in (0,\infty)$. Since the solution to the Dirichlet problem for L is unique by the maximum principle, in particular $Lu = 0$ weakly in Ω only when $u = 0$, $-1/\sigma$ is not an eigenvalue of L_σ^{-1} and thus there exists a unique solution $u \in W_0^{1,2}(\Omega)$ to $Lu = F$ weakly in Ω for every $F \in W_0^{1,2}(\Omega)^*$. □

The spectral theory for L_σ^{-1}, we obtain the Fredholm alternative for equations in divergence form:

Theorem 4 (Fredholm alternative)**.** *Let*

$$Lu = D_i(a^{ij} D_j u + b^i u) + c^j D_j u + du \text{ in } \Omega$$

for $u \in W^{1,2}(\Omega)$, where $a^{ij}, b^i, c^j, d \in L^\infty(\Omega)$ satisfying (4) and (5). There exists a countable, discrete set $\Sigma \subset \mathbb{R}$ such that

(a) if $\lambda \notin \Sigma$, the Dirichlet problem, $Lu + \lambda u = D_i f^i + g$ in Ω, $u = \varphi$ on $\partial\Omega$, has a unique solution $u \in W^{1,2}(\Omega)$ for all $f^i, g \in L^2(\Omega)$ and $\varphi \in W^{1,2}(\Omega)$, and

(b) if $\lambda \in \Sigma$, the homogeneous problem, $Lu + \lambda u = 0$ in Ω, $u = 0$ on $\partial\Omega$, has a finite dimensional subspace of nontrivial solutions $u \in W_0^{1,2}(\Omega)$. We call λ a Dirichlet eigenvalue of L.

(Note that some books, for example Gilbarg and Trudinger, define Σ as the set of λ such that there is a nontrivial solution $u \in W_0^{1,2}(\Omega)$ to $Lu - \lambda u = 0$ in Ω.)

4 Regularity theory

Theorem 5 ($W^{2,2}$ Interior Regularity)**.** *Let Ω be an open set in $\mathbb{R}^n$. Suppose $u \in W^{1,2}(\Omega)$ satisfies*

$$Lu = D_i(a^{ij}D_ju + b^iu) + c^jD_ju + du = f \text{ weakly in } \Omega$$

for an elliptic operator L with coefficients $a^{ij}, b^i \in C^{0,1}(\Omega)$ and $c^j, d \in L^\infty_{loc}(\Omega)$ and a function f with $f \in L^2_{loc}(\Omega)$. By elliptic, we just require that $a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2$ for some $\lambda \in (0,\infty)$. Then $u \in W^{2,2}_{loc}(\Omega)$ with

$$\|u\|_{W^{2,2}(\Omega')} \leq C(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{L^2(\Omega)})$$

for every $\Omega' \subset\subset \Omega$ for some constant $C = C(n, L, \Omega', \Omega) \in (0,\infty)$.

Theorem 6 ($W^{2+k,2}$ Interior Regularity for $k \geq 1$)**.** *Let $k \geq 1$ be an integer. Let Ω be an open set in $\mathbb{R}^n$. Suppose $u \in W^{1,2}(\Omega)$ satisfies*

$$Lu = D_i(a^{ij}D_ju + b^iu) + c^jD_ju + du = f \text{ weakly in } \Omega$$

for an elliptic operator L with coefficients $a^{ij}, b^i \in C^{k,1}(\Omega)$ and $c^j, d \in C^{k-1,1}(\Omega)$ and $f \in W^{k,2}_{loc}(\Omega)$. Then $u \in W^{k+2,2}_{loc}(\Omega)$ with

$$\|u\|_{W^{k+2,2}(\Omega')} \leq C(\|u\|_{W^{1,2}(\Omega)} + \|f\|_{W^{k,2}(\Omega)})$$

if $k \geq 1$ for every $\Omega' \subset\subset \Omega$ for some constant $C = C(n, k, L, \Omega', \Omega) \in (0,\infty)$.

Moreover, if $Lu = f$ in Ω for some elliptic operator L with coefficients $a^{ij}, b^i, c^j, d \in C^\infty(\Omega)$ and some $f \in C^\infty(\Omega)$, then by the Sobolev embedding theorem $u \in C^\infty(\Omega)$.

The proof of interior regularity follows more or less from a difference quotient argument like before using induction on k and energy estimates in place of the Schauder estimates in the case $k = 0$. However, we need to establish that the obvious difference quotient operator

$$\delta_{l,h}u(x) = \frac{u(x + he_l) - u(x)}{h}, \tag{20}$$

where $h \neq 0$ and $l = 1, \ldots, n$, has the correct properties in the case that u is a Sobolev function. We also need to be careful since $\delta_{l,h}f$ is not necessarily bounded locally in $W^{k,2}$ for $f \in W^{k,2}_{\text{loc}}(\Omega)$.

Lemma 2. *Let $u \in W^{1,p}(\Omega)$ for $1 \leq p < \infty$. Then $\delta_{l,h}u \in L^p(\Omega')$ for any $\Omega' \subset\subset \Omega$ with $\operatorname{dist}(\Omega', \partial\Omega) > h$ and*

$$\|\delta_{l,h}u\|_{L^p(\Omega')} \leq \|D_lu\|_{L^p(\Omega)}.$$

Proof. Since $C^\infty(\Omega)$ is dense in $W^{1,p}(\Omega)$ (see Gillbarg and Trudinger Theorem 7.9), it suffices to consider $u \in C^\infty(\Omega) \cap W^{1,p}(\Omega)$. We compute

$$\begin{aligned}
\int_{\Omega'} |\delta_{l,h}u(x)|^p dx &= \int_{\Omega'} \left| \frac{1}{h}\int_0^h D_lu(x + te_l)dt \right|^p dx && \text{(by the fundamental theorem of calculus)} \\
&\leq \int_{\Omega'} \frac{1}{h}\int_0^h |D_lu(x + te_l)|^p dtdx && \text{(by Hölder's inequality)} \\
&\leq \frac{1}{h}\int_0^h \int_{\Omega'} |D_lu(x + te_l)|^p dxdt && \text{(by Tonelli's theorem / Fubini's theorem)} \\
&\leq \frac{1}{h}\int_0^h \int_{\Omega} |D_lu(y)|^p dydt && \text{(by letting } y = x + te_l) \\
&\leq \int_{\Omega} |D_lu(y)|^p dy.
\end{aligned}$$

□

Lemma 3. *Let $u \in L^p(\Omega)$ for $1 < p < \infty$ and suppose*

$$\sup_{0<|h|<h_0} \|\delta_{l,h} u\|_{L^p(\Omega')} < \infty \tag{21}$$

for every $\Omega' \subset\subset \Omega$ and $h_0 = \operatorname{dist}(\Omega', \partial\Omega)$. Then the weak derivative $D_l u \in L^p_{loc}(\Omega)$ exists and

$$\|D_l u\|_{L^p(\Omega')} \le \sup_{0<|h|<h_0} \|\delta_{l,h} u\|_{L^p(\Omega')}.$$

for every $\Omega' \subset\subset \Omega$ and $h_0 = \operatorname{dist}(\Omega', \partial\Omega)$.

Proof. Example sheet. □

Proof of $W^{2,2}$ Interior Regularity. Recall that

$$\int_\Omega ((a^{ij} D_j u + b^i u) D_i \zeta - (c^j D_j u + du)\zeta) = -\int_\Omega f\zeta$$

for every $\zeta \in W_0^{1,2}(\Omega)$. Since $b^i \in C^{0,1}(\Omega)$ and $u \in W^{1,2}(\Omega)$, by integration by parts,

$$\int_\Omega a^{ij} D_j u D_i \zeta = \int_\Omega ((b^i + c^i) D_i u + (D_i b^i + d)u - f)\zeta = \int_\Omega g\zeta$$

for every $\zeta \in W_0^{1,2}(\Omega)$, where $g = (b^i + c^i) D_i u + (D_i b^i + d)u - f$. Replace ζ by $\delta_{l,-h}\zeta$ to get

$$\begin{aligned}\int_\Omega a^{ij}(x + he_l)\, D_j \delta_{l,h} u(x)\, D_i \zeta(x)\, dx &= \int_\Omega \delta_{l,h}(a^{ij} D_j u)\, D_i \zeta - \int_\Omega \delta_{l,h} a^{ij}\, D_j u\, D_i \zeta \\ &= -\int_\Omega a^{ij}\, D_j u\, D_i \delta_{l,-h}\zeta - \int_\Omega \delta_{l,h} a^{ij}\, D_j u\, D_i \zeta \\ &= -\int_\Omega g\, \delta_{l,-h}\zeta - \int_\Omega \delta_{l,h} a^{ij}\, D_j u\, D_i \zeta\end{aligned}$$

for every $\zeta \in W_0^{1,2}(\Omega)$. Using (5), Cauchy-Schwartz, and the properties of difference quotients,

$$\begin{aligned}\left|\int_\Omega a^{ij}(x + he_l)\, D_j \delta_{l,h} u(x)\, D_i \zeta(x)\, dx\right| &\le \|g\|_{L^2(\Omega)} \|\delta_{l,-h}\zeta\|_{L^2(\Omega)} + C\|Du\|_{L^2(\Omega)} \|D\zeta\|_{L^2(\Omega)} \\ &\le C(\|g\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)}) \|D\zeta\|_{L^2(\Omega)}\end{aligned}$$

for every $\zeta \in W_0^{1,2}(\Omega)$, where $\|g\|_{L^2}$ and $\|Du\|_{L^2}$ are L^2 norms over the support of ζ, provided $|h|$ is less than the distance of the support of ζ to $\partial\Omega$.

Choose $\Omega' \subset\subset \Omega$ and $\eta \in C_0^1(\Omega)$ satisfying $0 \le \eta \le 1$ on Ω, $\eta = 1$ on Ω' and $|D\eta| \le 2/d(\Omega', \Omega)$. Then, taking $\zeta = \eta^2 \delta_{l,h} u$, for h sufficiently small (depending on the support of η) the previous computation, the ellipticity assumption, and the assumption that $|\eta| \le 1$ imply that

$$\begin{aligned}\lambda \int_\Omega \eta^2 |D\delta_{l,h} u|^2 dx &\le \int_\Omega a^{ij}(x + he_l) \eta^2 D_j \delta_{l,h} u(x) D_i \delta_{l,h} u(x) dx \\ &= \int_\Omega a^{ij}(x + he_l) D_j \delta_{l,h} u(x) D_i(\eta^2 \delta_{l,h} u)(x) dx \\ &\quad - 2\int_\Omega a^{ij}(x + he_l) \eta \delta_{l,h} u(x) D_j \delta_{l,h} u(x) D_i \eta(x) dx \\ &\le C(1 + \|g\|_{L^2(\Omega)} + \|Du\|_{L^2(\Omega)})(\|\eta D\delta_{l,h} u\|_{L^2(\Omega)} + \|(\delta_{l,h} u) D\eta\|_{L^2(\Omega)})\end{aligned}$$

Absorbing the $D\delta_{l,h}u$ terms into the right hand side and using the above lemmas to relate the discrete difference quotient to the derivative, we find that

$$\frac{\lambda}{2}\int_{\Omega'}|D\delta_{l,h}u|^2dx \leq C\int_{\Omega}(|u|^2+|Du|^2+|g|^2)dx.$$

Thus, because $\delta_{l,h}Du$ is uniformly bounded in $L^2(\Omega')$, we see that $u \in W^{2,2}(\Omega)$. Letting $h \to 0$, the estimate follows. □

Theorem 7 ($W^{k+2,2}$ Global Regularity)**.** *Let $k \geq 1$ be an integer. Let Ω be a C^{k+2} domain in $\mathbb{R}^n$. Suppose $u \in W^{1,2}(\Omega)$ satisfies*

$$Lu = D_i(a^{ij}D_ju + b^iu) + c^jD_ju + du = f \textit{ weakly in } \Omega$$

for an elliptic operator L with coefficients $a^{ij}, b^i \in C^{k,1}(\overline{\Omega})$, $c^j, d \in C^{k-1,1}(\overline{\Omega})$, $f \in W^{k,2}(\Omega)$, and $\varphi \in W^{k+2,2}(\Omega)$. Then $u \in W^{k+2,2}(\Omega)$ with

$$\|u\|_{W^{2,2}(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} + \|\varphi\|_{W^{2,2}(\Omega)})$$

if $k=0$ and

$$\|u\|_{W^{k+2,2}(\Omega)} \leq C(\|u\|_{L^2(\Omega)} + \|f\|_{W^{k,2}(\Omega)} + \|\varphi\|_{W^{k+2,2}(\Omega)})$$

if $k \geq 1$ for some constant $C = C(n,k,L,\Omega) \in (0,\infty)$.

Moreover, if $Lu = f$ in Ω and $u = \varphi$ on $\partial\Omega$ for some elliptic operator L with coefficients $a^{ij}, b^i, c^j, d \in C^\infty(\overline{\Omega})$ and some $f, \varphi \in C^\infty(\overline{\Omega})$, then by the Sobolev embedding theorem $u \in C^\infty(\overline{\Omega})$.

The proof is fairly standard. We proceed by induction on k. To prove the $W^{2,2}$ regularity near a point $y \in \partial\Omega$, we can reduce to the case where $\varphi = 0$ by replacing u with $u - \varphi$ and we can replace to the case where $y = 0$ and $\Omega \cap B_R(0) = B_R^+$ by using a C^1 diffeomorphism. By applying the difference quotient argument in the proof of $W^{2,2}$ interior regularity, using the fact that $\eta^2 u \in W_0^{1,2}(\Omega)$ when $\eta \in C_c^\infty(\mathbb{R}^n)$ is the cutoff function such that $\eta = 1$ on $B_{R/2}$, $\eta = 0$ on $\mathbb{R}^n \setminus B_R$, and $|D\eta| \leq 3/R$, we can show that $D_lu \in W^{1,2}(B_{R/2}^+)$ for $l = 1, 2, \ldots, n-1$. By the differential equation,

$$a^{nn}D_{nn}u = f - \sum_{(i,j)\neq(n,n)} a^{ij}D_{ij}u - \sum_{j=1}^n\left(\sum_{i=1}^n D_ia^{ij} + b^j + c^j\right)D_ju - \left(\sum_{i=1}^n D_ib^i + d\right)u \in L^2(B_{R/2}^+),$$

completing the proof that $u \in W^{2,2}(B_{R/2}^+)$.

Note that as an immediate consequence of the existence theory and global regularity, whenever Ω is a C^∞ domain, $a^{ij}, b^i, c^i, d \in C^\infty(\overline{\Omega})$ satisfy (4), (5), and (7), and $f \in C^\infty(\overline{\Omega})$, there exists a unique function $u \in C^\infty(\Omega)$ such that $Lu = f$ weakly in Ω and $u = \varphi$ on $\partial\Omega$. As was discussed previously, this implies that $Lu = f$ pointwise in Ω and $u = \varphi$ pointwise on $\partial\Omega$.

Note that by using the scaling argument from the proof of the $C^{2,\mu}$ Schauder estimates for classical solutions, we also get $C^{1,\mu}$ Schauder estimates on weak solutions to elliptic equations in divergence form. For example:

Theorem 8 (Interior $C^{1,\mu}$ Estimate)**.** *Let $\mu \in (0,1)$. Suppose $u \in C^{1,\mu}(\overline{B_R(x_0)})$ satisfies*

$$Lu = D_i(a^{ij}D_ju + b^iu) + c^jD_ju + du = D_if + g \textit{ weakly in } B_R(x_0)$$

where

$$a^{ij}(x)\xi_i\xi_j \geq \lambda|\xi|^2 \text{ for a.e. } x \in B_R(x_0) \text{ and all } \xi \in \mathbb{R}^n$$

for some constant $\lambda > 0$ *and* $a^{ij}, b^i \in C^{0,\mu}(\overline{B_R(x_0)})$ *and* $c^i, d \in C^0(\overline{B_R(x_0)})$ *such that*

$$|a^{ij}|'_{0,\mu;B_R(x_0)} + R|b^i|'_{0,\mu;B_R(x_0)} + R|c^i|_{0;B_R(x_0)} + R^2|d|_{0;B_R(x_0)} \leq\leq \nu$$

for some constant $\nu \in (0,\infty)$ *and* $f^i \in C^{0,\mu}(\overline{B_R(x_0)})$ *and* $g \in C^0(\overline{B_R(x_0)})$*. Then*

$$|u|'_{1,\mu;B_{R/2}(x_0)} \leq C(\|u\|_{L^2(B_R(x_0))} + R^{1+\mu}[f]_{\mu;B_R(x_0)} + R^2|g|_{0;B_R(x_0)})$$

for some constant $C = C(n,\lambda,\nu) \in (0,\infty)$*.*

References: Gilbarg and Trudinger, Chapter 8.